and

$$[L] = [F]^*[D][F] \quad (p \times p \text{ matrix})$$

Thus we have transformed Eq. (1) to standard first-order differential form from which standard eigenvalue routines can be employed for solution.

The roots of the original system are the eigenvalues ($[\lambda]$) of $[L]$. These values and vectors satisfy the following relationship:

$$\begin{aligned} [L][Z] &= [Z][\lambda] \\ [L]^t[W] &= [W][\lambda] \end{aligned} \tag{10}$$

where

$$[W]^t[Z] = [I]$$

For the bi-orthogonal system described previously a matrix $[Z]$ cannot be found such that $[Z]^{-1}[L][Z]$ is diagonal, if $[L]$ contains multiple roots and is defective. If $[L]$ is a matrix of order p it is called defective[1] if it has fewer than p independent eigenvectors. This situation arises for undamped rigid body motion. To illustrate, consider the following undamped single degree-of-freedom system with a unit mass:

$$[I]\begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} + [L]\begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} = \begin{Bmatrix} P(t) \\ 0 \end{Bmatrix} \tag{11}$$

where

$$[L] = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

Here, the $[L]$ matrix has two eigenvalues $\lambda_1 = \lambda_2 = 0$, but only one independent vector $\{0,1\}$. Undamped rigid body motion can be handled separately; for linear systems it suffices to say that its effect can be superimposed on the solution described herein. Therefore the matrices $[Z]$ and $[W]$ for the bi-orthogonal nondefective portion of the system will not, in general, be square. Also it should be clear that if $[A_1]$, $[A_2]$, and $[A_3]$ contain real coefficients only, the nonreal λ's will appear as complex conjugate pairs.

Let

$$\{s_1\} = [Z]\{\xi\} \tag{12}$$

and premultiply Eq. (9) by $[W]^t$ to obtain the desired decoupled equations of motion;

$$\{\dot{\xi}\} + [\lambda]\{\xi\} = [T]\{P(t)\} \tag{13}$$

where

$$[T] = [W]^t[\gamma]$$

From the previous developments one can obtain the following transformations:

$$\begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} = [Y]\{\xi\} + [\mathbf{v}]\begin{bmatrix} 0 \\ D_{22}{}^{-1}\alpha \end{bmatrix}\{P(t)\} \tag{14}$$

where

$$[Y] = [\mathbf{v}]\begin{bmatrix} d^{-1} & 0 \\ 0 & I \end{bmatrix}[F][Z]$$

and

$$\{\xi\} = [X]^t\begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} \tag{15}$$

where

$$[X]^t = [W]^t[\diagdown d \diagdown|0][\mathbf{v}]^*$$

The solution of Eq. (1) for the nondefective portion of the system is contained in the real part of Eq. (14). This is, of course, a superposition of the decoupled systems $\{Y_j\}\xi_j$'s.

The solution of the ξ_j's [Eq. (13)] is subject to the initial condition

$$\xi_{jo} = \{x_j\}^t\begin{Bmatrix} \dot{u}_o \\ u_o \end{Bmatrix}$$

[Eq. (15)], where $\{\dot{u}_o\}$ and $\{u_o\}$ are initial velocities and displacements, respectively.

Reference

[1] J. H. Wilkinson, *The Algebraic Eigenvalue Problem*, Oxford University Press, London, 1965, Chap. 1.

Effect of Initial Body Motion on Transient Amplification

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Nomenclature

$C_{m\alpha}$	= aerodynamic moment coefficient slope
d	= reference length
I	= $Ix = Iy$
I_x, I_y, I_z	= moment of inertia about pitch, yaw, and roll axis
K_2	= stability parameter = $\rho_0 Sd(-C_{m\alpha})/2I\beta^2 \sin^2\gamma$
K_3	= spin parameter = $p_0(1 + \lambda)/\beta V \sin\gamma$
p_0	= spin rate = const
p_{cr}	= critical frequency $\approx [\rho V^2 Sd(-C_{m\alpha})/2(I - I_z)]^{1/2}$
q_E, r_E	= components of initial angular velocity along the pitch and yaw axis respectively
S	= reference area
t	= time
V	= vehicle velocity = const
y	= altitude
α	= complex angle of attack = $\theta_e{}^{-i\psi}$
α_H	= see Eq. (5)
α_T	= trim angle of attack, Eq. (8)
β	= constant in density-altitude relation
γ	= flight path angle from local horizontal
ζ	= $K_2{}^{1/2}e^{-\beta y/2}$
ϕ, θ, ψ	= Eulerian angles (see Fig. 1)
λ	= $(I_Z/I) - 1$
ρ	= $\rho_0 e^{-\beta y}$ = air density
ρ_0	= sea level density
ω_E	= initial tumble rate = $q_E + ir_E$
1, i	= unit vectors (see Fig. 1)
$(\dot{\ })$	= d/dt
$(\ )'$	= $d/d\zeta$

Introduction

THE effect of initial body attitude and rates on the transient amplification of a static trim angle of attack is of interest in evaluating maximum normal loads during reentry. The present analysis provides an indication of the effect of initial body motion on the magnitude of the envelope value of angle of attack throughout the trajectory including the altitude at which the vehicle encounters the maximum transient amplification of a static trim[1] which can be variable with altitude.

Development of General Equations

The basic equation used in the present analysis to define the vehicle rotational motion is that of Leon[2] and has been derived assuming negligible aerodynamic damping, constant velocity, flight path angle and spin rate, zero products of inertia, small angles of attack, and exponential density variation.

The Eulerian angles describing the angular orientation of the vehicle is shown in Fig. 1. A static trim angle of attack

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has been added to the equation developed in Ref. 2 and the vehicle rotation motion is defined by

$$\ddot{\alpha} + i(1 - \lambda)p_0\dot{\alpha} + \{\lambda p_0^2 + [\rho V^2 Sd(-C_{m\alpha})/2I]\}\alpha = -[\rho V^2 Sd(-C_{m\alpha})/2I]\alpha_T(t) \quad (1)$$

The trim angle of attack $\alpha_T(t)$ can be variable within certain restrictions which will be discussed below.

The primary region of interest in the present discussion is near the altitude where $p_0 \approx p_{cr}$. Using the change of variable,

$$\zeta^2 = K_2 e^{-\beta y} \quad (2)$$

the altitude at which $p_0 \approx p_{cr}$ is defined by

$$\zeta_{cr} = K_3(-\lambda)^{1/2}/(1 + \lambda) \quad (3)$$

and Eq. (1) becomes

$$\zeta^2\alpha'' + \zeta\{1 + i2K_3[(1 - \lambda)/(1 + \lambda)]\}\alpha' + (-4\zeta_{cr}^2 + 4\zeta^2)\alpha = -4\zeta^2\alpha_T(\zeta) \quad (4)$$

which is a form of the Bessel equation as shown in Ref. 3. The homogeneous solution of Eq. (4) is

$$\alpha_H(\zeta) = \zeta^{-iK_3(1-\lambda)/(1+\lambda)}\{AJ_{ik_3}(2\zeta) + BJ_{-ik_3}(2\zeta)\} \quad (5)$$

where A and B are constants to be defined by the initial conditions. The complete solution[3] of Eq. (5) is

$$\alpha(\zeta) = i\frac{2\pi}{\sinh \pi K_3}\zeta^{-ik_3[(1-\lambda)/(1+\lambda)]} \times \Big\{J_{ik_3}(2\zeta)\int_0^\zeta \alpha_T(\zeta)\zeta^{1+iK_3[(1-\lambda)/(1+\lambda)]}J_{-ik_3}(2\zeta)d\zeta - J_{-ik_3}(2\zeta)\int_0^\zeta \alpha_T(\zeta)\zeta^{1+ik_3[(1-\lambda)/(1+\lambda)]}J_{ik_3}(2\zeta)d\zeta\Big\} + \alpha_H(\zeta) \quad (6)$$

High-Altitude Solution

At high altitudes as $y \to \infty$, $\zeta \to 0$, the limiting forms of Bessel functions for small arguments are

$$J_{\pm ik_3}(2\zeta) \sim \zeta^{\pm ik_3}/\Gamma(1 \pm ik_3) = \zeta^{\pm ik_3}\Gamma(1 \mp ik_3)\sinh \pi K_3/\pi K_3 \quad (7)$$

At this point, a relation defining the trim must be assumed. Undoubtedly many equations which relate trim angle of attack and altitude could be devised. The relation used here is

$$\alpha_T(\zeta) = \sum_{n=0}^{N} \alpha_{T_n}\zeta^{b_n} \quad (8)$$

which appears adequate, generally as well as tractable, analytically.

Incorporating Eqs. (7) and (8) into Eq. (6) and integrating, the angle of attack at high altitudes is

$$\alpha_{\zeta\to 0}(\zeta) = -\sum_{n=0}^{N}(K_n\alpha_{T_n}\zeta^{b_n+2}) + \frac{A}{\Gamma(1 + iK_3)} \times \zeta^{i[2K_3\lambda/(1+\lambda)]} + \frac{B}{\Gamma(1 - iK_3)}\zeta^{-i[2K_3/(1+\lambda)]} \quad (9)$$

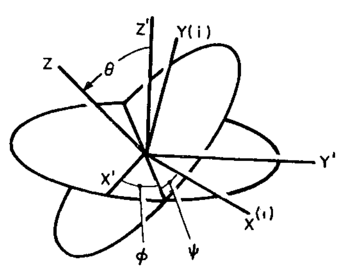

Fig. 1 Coordinate system.

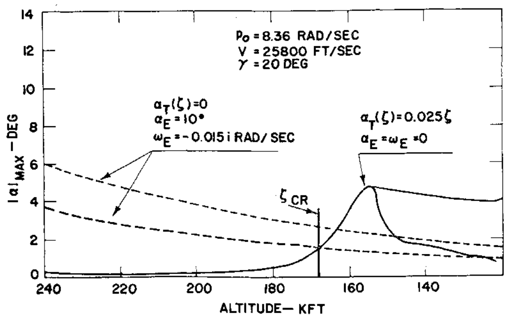

Fig. 2 Angle-of-attack contributions.

where

$$K_n = 1\Big/\left[\left(\frac{b_n + 2}{2}\right)^2 + \frac{\lambda K_3^2}{(1 + \lambda)^2} + iK_3 \times \left(\frac{1 - \lambda}{1 + \lambda}\right)\left(\frac{b_n + 2}{2}\right)\right] \quad (10)$$

The trim contribution, Eq. (9) goes to zero for all values of

$$b_n > -2 \quad (11)$$

imposing a further restriction on the definition of $\alpha_T(\zeta)$.

The resulting high-altitude solution, for $b_n > -2$, therefore becomes the homogeneous solution and the constants of integration are obtained from Ref. 2, which in the present notation are

$$A = [-i\omega_E/p_0(1 + \lambda)]\Gamma(1 + iK_3)K_2^{-i[\lambda K_3/(1+\lambda)]} \quad (12)$$

and

$$B = [\alpha_E + i\omega_E/p_0(1 + \lambda)]\Gamma(1 - iK_3)K_2^{i[K_3/(1+\lambda)]} \quad (13)$$

Nature of the Solution

Rewriting Eq. (6) and taking the absolute value to obtain the envelope value of angle of attack, the inequality

$$|\alpha(\zeta)|_U \leq |\alpha_f(\zeta)| + |\alpha_H(\zeta)| \quad (14)$$

for the upper envelope and

$$|\alpha(\zeta)|_L \leq ||\alpha_f(\zeta) - |\alpha_H(\zeta)|| \quad (15)$$

for the lower envelope is obtained.

$\alpha_f(\zeta)$, the first two terms of Eq. (6), is dependent on $\alpha_T(\zeta)$ and independent of α_E and ω_E. $\alpha_H(\zeta)$ is a function of α_E, and ω_E, and independent of $\alpha_T(\zeta)$. Equations (14) and (15) show that the trim contribution $\alpha_f(\zeta)$ and the initial conditions contribution $\alpha_H(\zeta)$ can be evaluated independently and combined to obtain bounds on $|\alpha|$.

A simplified illustration is shown in Fig. 2 and 3 using 6 degree-of-freedom results. The solid curves of Fig. 2 represent the upper and lower envelopes of angle of attack of a body with trim and the dashed curves represent the upper

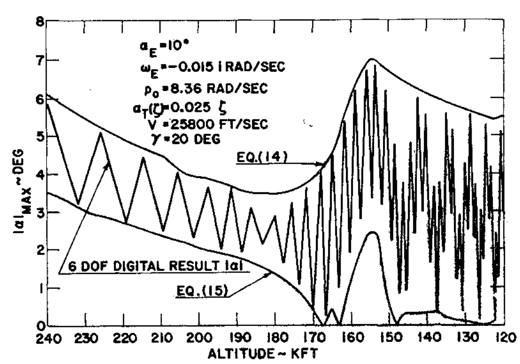

Fig. 3 Envelope of angle of attack.

and lower envelopes of a body with an initial attitude and rate.

Using the data of Fig. 2 in Eq. (14) and (15), predictions of the upper and lower bounds of angle of attack for a body with trim and initial attitude and rate are shown in Fig. 3 along with 6 degree-of-freedom results. The upper bound of Fig. 3 was computed using the sum of the two maximums from Fig. 2. The lower bound of Fig. 3 was computed using the envelope values from Fig. 2 which resulted in the smallest magnitude. The comparison Fig. 3, shows that the sum of individual contributions bound the 6 degree-of-freedom computer results.

The analysis indicates that the individual effects of trim and initial conditions can be combined in a straightforward manner for a range of conditions including a variable trim history to obtain bounds on the envelope of angle of attack.

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[2] Leon, H. I., "Angle of Attack Convergence of a Spinning Missile Descending Through the Atmosphere," *Journal of the Aerospace Sciences*, Vol. 25, No. 8, Aug. 1958.

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Derivation of the Shell Compatibility Equations

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Introduction

The strain compatibility equations have been treated in numerous works on thin shell theory. The first attempt to derive the compatibility equations in shell theory is attributed to Odqvist in 1937.[1]

The first complete derivation of the strain compatibility equations for linear shell theory was given by Gol'denveizer.[2] In his work, Gol'denveizer utilized the condition that the shell displacements must be independent of the path in shell space when calculated from the linearized strain-displacement relations.

Derivations of the compatibility equations in linear and nonlinear shell theory also have utilized the Gauss and Mainardi-Codazzi conditions as a basis.[3–5] These geometric conditions can be written in terms of the membrane and bending strains. Satisfying the resulting set of differential equations insures that the strain measures are analytic functions of the shell reference surface displacements.

The purpose of this Note is to re-examine the shell strain compatibility equations from a physical aspect. The derivation is based on the requirement that the shell membrane and bending strains must be independent of the natural paths from point to point on the reference surface. The natural paths are taken to the shell reference surface coordinates.

The derivation given in this paper is restricted to a discussion of the strain compatibility conditions on an arbitrary reference surface in the shell continuum, thus implying the assumption of the Kirchhoff hypotheses. As a consequence, the state of strain in the surface can be described exclusively in terms of the first and second fundamental forms of deformed and undeformed surfaces. The derivation does not rely on any form of the Gauss and Mainardi-Codazzi equations as a basis. Instead, these equations arise as a natural consequence of the arguments presented herein.

Derivation of Compatibility Equations

The reference surface S of a shell is described by curvilinear coordinates ξ^α ($\alpha = 1, 2$). As a result of external forces, the shell reference surface deforms to S^*. The relationship between S and S^* can be conveniently described in terms of the displacements, i.e.,

$$X = x^i + \phi^i \tag{1}$$

In what follows, English indices extend over the range 1, 2, 3, while Greek indices extend over the range 1, 2. Equation (1) describes the position of a point Q^* on S^* in terms of the position vector x^i of a point Q on S and the displacements ϕ^i. The displacement vector ϕ^i and position vector x^i are functions of the coordinates ξ^α.

The vectors tangent and normal to the coordinate curves at point Q^* and S^* are given, respectively, by

$$dX^i = X_\alpha{}^i d\xi^\alpha \tag{2}$$

$$N_i = \tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{ijk}X_\alpha{}^i X_\beta{}^k \tag{3}$$

The quantities, $\epsilon^{\alpha\beta}$ and ϵ_{ijk}, in Eq. (3), are the permutation symbols of the deformed and undeformed surface, respectively.[6] The coefficients of the first and second fundamental tensors of the surface S^*, respectively, are defined as

$$G_{\alpha\beta} = X_\alpha{}^i X_\beta{}^i \tag{4a}$$

$$B_{\alpha\beta} = X^i{}_{\alpha|\beta}N^i = -X_\alpha{}^i N^i{}_{|\beta} \tag{4b}$$

The vertical bar in Eq. (4) denotes covariant differentiation with respect to the metric of the surface S^*. Equations (4a) and (4b) have been used as measures of the stretching and bending of the shell reference surface, respectively. Thus

$$2e_{\alpha\beta} = G_{\alpha\beta} - g_{\alpha\beta} \tag{5a}$$

$$\rho_{\alpha\beta} = B_{\alpha\beta} - b_{\alpha\beta} \tag{5b}$$

where $g_{\alpha\beta}$ and $b_{\alpha\beta}$ are the first and second fundamental forms for the undeformed reference surfaces.

The base vectors at point Q^* on S^* are considered here to be functions of ξ^α belonging to class C^2, and it is desired to express the base vectors at point T^* on S^* in terms of the base vectors at Q^*. The coordinates of the shell surface makes it possible to choose two obvious paths from point Q^* to point T^*, as shown in Fig. 1. The base vectors at T^* that result from following path $Q^* - U^* - T^*$ are denoted by $\tilde{X}_\alpha{}^i$ and $\tilde{N}_i$. Those base vectors at T^* resulting from following path $Q^* - V^* - T^*$ are denoted by $\hat{X}_\alpha{}^i$ and $\hat{N}_i$. The membrane and bending strains at T^* for the two sets of base

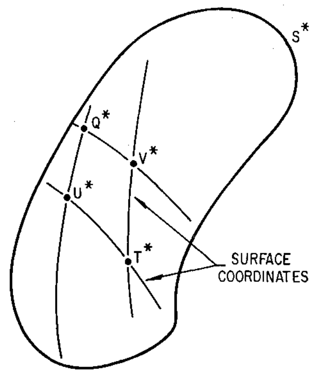

Fig. 1 Natural paths on the deformed surface.

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